NONLINEAR OSCILLATIONS OF COMBUSTION VELOCITY OF POWDER

B. V. Novozhilov

The combustion of powder, where the surface temperature T_s depends on the pressure p and the initial temperature T_0, is studied under the condition of sinusoidally varying pressure. The nonlinearity of the heat conduction equation together with the dependence of the combustion velocity u and the surface temperature on both the pressure and initial temperature affects the zeroth harmonic and gives rise to higher harmonics in the combustion velocity and the temperature of the powder. The present paper considers the case of nonlinear resonance, when the frequency of the pressure fluctuations is close to the natural vibration frequency of the powder. It has been shown that an autonomous oscillatory regime of combustion is possible under constant pressure.

1. FORMULATION OF PROBLEM AND BASIC EQUATIONS

It was shown in [1] that nonstationary processes of powder combustion with variable surface temperature can be studied with the same method as that used by Zel'dovich [2,3] for the case of constant temperature T_S. If all inertias except the heat transfer in the condensed phase are neglected, it can be shown that the surface temperature T_S of the powder, together with the combustion velocity u during nonstationary combustion are determined by the instantaneous values of the pressure and internal temperature gradient at the powder surface f. The relations $u(f, p)$ and $T_S(f, p)$ can be obtained from the laws of steady-state combustion $u^\circ(T_0, p)$ and $T_S^\circ(T_0, p)$ by replacing the initial temperature T_0 using the following expression in terms of u, T_S, and f

$$\varkappa f^\circ = u^\circ (T_s^\circ - T_0) \qquad (1.1)$$

which is valid for the stationary regime ($\varkappa$ is the thermal conductivity of the powder).

The linearized problem of powder combustion under sinusoidally varying pressure was studied in [4]. In the present paper we study nonlinear effects including nonlinear resonance and autonomous oscillations.

Let p° be the mean pressure and u°, T_S° be the corresponding combustion velocity and surface temperature in the steady-state combustion at pressure p°.

Now we define the dimensionless variables

$$\theta = \frac{T - T_0}{T_s^\circ - T_0}, \qquad \xi = \frac{u^\circ}{\varkappa} x, \qquad \tau = \frac{(u^\circ)^2}{\varkappa} t,$$

$$v = \frac{u}{u^\circ}, \qquad \eta = \frac{p}{p^\circ}, \qquad \varphi = \frac{\partial \theta}{\partial \xi}\Big|_{\xi=0} \qquad (1.2)$$

where x is the coordinate (x < 0), and t is the time. In the inertial region, i. e. , the heated layer of the condensed phase, we have the following heat conduction equation,

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} - v \frac{\partial \theta}{\partial \xi} \qquad (1.3)$$

together with the boundary conditions

$$\theta |_{\xi=0} = \vartheta, \qquad \theta |_{\xi \to -\infty} = 0 \qquad \left(\vartheta = \frac{T_s - T_0}{T_s^\circ - T_0} \right). \quad (1.4)$$

The system of equations (1.3), (1.4) becomes complete when supplemented by the following relations for the combustion velocity and surface temperature in terms of the pressure and gradient

$$v = v (\varphi, \eta), \qquad \vartheta = \vartheta (\varphi, \eta) \qquad (1.5)$$

together with the expression for the pressure which will be chosen to have the cosinusoidal form

$$\eta = 1 + h (\cos \gamma\tau + \sin \gamma\tau) \qquad (1.6)$$

where $(2)^{1/2} h = H$ is the pressure amplitude.

The system (1.3)–(1.6) theoretically determines the time dependence of combustion velocity and powder temperature, but it is not possible to solve this system for arbitrary functions (1.5). However, it is possible to expand the solution in a series of powers of the small pressure amplitude.

2. COMBUSTION VELOCITY AND TEMPERATURE DISTRIBUTION: THIRD ORDER APPROXIMATION

A periodic force applied to a nonlinear vibratory system gives rise to oscillations at all harmonics. If the amplitude of the force is small, the correction to the constant component and the second harmonic is proportional to the square of the amplitude, and the first harmonic, as compared with the linear approximation, has a third order correction in terms of the amplitude (see [5, 6, 7]).

The steady-state of combustion at pressure $\eta = 1$ is described by the solution

$$\theta = e^\xi, \quad v = 1, \quad \vartheta = 1, \quad \varphi = 1 . \qquad (2.1)$$

When the pressure varies according to (1.6), the higher order approximation will be sought in the form

$$v (\tau) = 1 + v_{1c} \cos \gamma\tau +$$
$$+ v_{1s} \sin \gamma\tau + w_2 + v_{2c} \cos 2\gamma\tau + v_{2s} \sin 2\gamma\tau, \quad (2.2)$$

$$\vartheta (\tau) = 1 + \vartheta_{1c} \cos \gamma\tau +$$
$$+ \vartheta_{1s} \sin \gamma\tau + \psi_2 + \vartheta_{2c} \cos 2\gamma\tau + \vartheta_{2s} \sin 2\gamma\tau, \quad (2.3)$$

$$\varphi (\tau) = 1 + \varphi_{1c} \cos \gamma\tau +$$
$$+ \varphi_{1s} \sin \gamma\tau + f_2 + \varphi_{2c} \cos 2\gamma\tau + \varphi_{2s} \sin 2\gamma\tau, \quad (2.4)$$

$$\theta (\xi_1\tau) = e^{(1+w_2)\xi} [1 + \theta_{1c}(\xi) \cos \gamma\tau +$$

$$+ \theta_{1s}(\xi) \sin \gamma\tau + \Psi_2(\xi) +$$
$$+ \theta_{2c}(\xi) \cos 2\gamma\tau + \theta_{2s} \sin 2\gamma\tau] \qquad (2.5)$$

We substitute expression (2.2) for the velocity and (2.5) for the temperature into the heat conduction equation, multiply the trigonometric functions, retaining only the zeroth, 1st, and 2nd harmonics, and then we equate coefficients of like trigonometric functions. This yields five ordinary differential equations for the unknowns θ_{1C}, θ_{1S}, Ψ_2, θ_{2C}, and θ_{2S}.

The complex notation

$$\theta_n = \theta_{ns} + i\theta_{nc}, \qquad v_n = v_{ns} + iv_{nc}$$
$$\varphi_n = \varphi_{ns} + i\varphi_{nc}, \qquad \vartheta_n = \vartheta_{ns} + i\vartheta_{nc} \qquad (n = 1,2) \quad (2.6)$$

enables us to reduce the five differential equations to the following two complex and one real differential equation

$$\theta_1'' + \theta_1' - i\gamma\theta_1 = v_1 - w_2\theta_1' +$$
$$+ v_1(w_2 + \Psi_2' + \Psi_2) + \tfrac{1}{2}i\bar{v}_1(\theta_2' + \theta_2) +$$
$$+ \tfrac{1}{2}i\,v_2(\bar\theta_1' + \theta_1), \qquad (2.7)$$

$$\Psi_2'' + \Psi_2' = \tfrac{1}{4}v_1(\bar\theta_1' + \bar\theta_1) + \tfrac{1}{4}\bar{v}_1(\theta_1' + \theta_1), \quad (2.8)$$

$$\theta_2'' + \theta_2' - 2i\gamma\theta_2 = v_2 - \tfrac{1}{2}iv_1(\theta_1' + \theta_1), \qquad (2.9)$$

where the prime denotes differentiation with respect to ξ, and the bar indicates a complex conjugate. In the linear approximation, (2.7) has the solution [4]

$$\theta_1 = \vartheta_1 + A_1(e^{z_1\xi} - 1), \qquad A_1 = \vartheta_1 - iv_1/\gamma \quad (2.10)$$

where

$$z_1 = \tfrac{1}{2}(\gamma/R_1 - 1) + iR_1,$$
$$R_1 = [\tfrac{1}{8}(\sqrt{16\gamma^2 + 1} - 1)]^{1/2}. \qquad (2.11)$$

It is worth noting that

$$z_1^2 + z_1 - i\gamma = 0, \qquad 4R_1^4 + R_1^2 - \gamma^2 = 0. \quad (2.12)$$

Having the expression for $\theta_1(\xi)$ in the first approximation, we can find $\Psi_2(\xi)$ and $\theta_2(\xi)$ in the second order approximation from equations (2.8), (2.9). The constant component is found to be

$$\Psi_2 = \psi_2 + \frac{i}{4\gamma}[\bar{A}_1v_1(1 + \bar{z}_1) \times$$
$$\times (e^{\bar{z}_1\xi} - 1) - A_1\bar{v}_1(1 + z_1)(e^{z_1\xi} - 1)]. \qquad (2.13)$$

The spatial distribution of the second harmonic of the temperature in the condensed phase is

$$\theta_2 = \vartheta_2 + \left(A_2 - \frac{iv_1^2}{4\gamma^2}\right) \times$$
$$\times (e^{z_2\xi} - 1) + \frac{v_1A_1}{2\gamma}(1 + z_1)(e^{z_1\xi} - e^{z_2\xi}) \qquad (2.14)$$

where

$$A_2 = \vartheta_2 - \frac{iv_2}{2\gamma}, \qquad z_2 = \frac{1}{2}\left(\frac{2\gamma}{R_2} - 1\right) + iR_2,$$
$$R_2 = \left[\frac{1}{8}(\sqrt{64\gamma^2 + 1} - 1)\right]^{1/2}$$

$$z_2^2 + z_2 - 2i\gamma = 0,$$
$$4R_2^4 + R_2^2 - 4\gamma^2 = 0. \qquad (2.15)$$

Finally, substituting (2.10), (2.13) into the right-hand side of (2.7) and solving this equation, we obtain the amplitude of the first harmonic with an accuracy up to and including third order terms as given below

$$\theta_1 = \vartheta_1 + \left(A_1 + A_3 - \frac{i\bar{v}_1v_2}{4\gamma_2}\right)(e^{z_1\xi} - 1) -$$
$$- \frac{A_1w_2z_1}{1 + 2z_1}\xi e^{z_1\xi} - \frac{A_1(1 + \bar{z}_1)}{4\gamma}\left[v_2 + \frac{v_1^2}{2\gamma}(1 + z_1)\right] \times$$
$$\times (e^{\bar{z}_1\xi} - e^{z_1\xi}) + \frac{v_1}{2\gamma}(1 + z_2)(e^{z_2\xi} - e^{z_1\xi}) \times$$
$$\times \left[A_2 - \frac{v_1A_1}{2\gamma}(1 + z_1) - \frac{i}{4\gamma^2}v_1^2\right],$$
$$A_3 = -\frac{i}{\gamma}v_1\left\{w_2 + \psi_2 - \frac{v_1\bar{v}_1}{8\gamma^2} -\right.$$
$$\left. - \frac{i}{4\gamma}[\bar{A}_1v_1(1 + \bar{z}_1) - A_1\bar{v}_1(1 + z_1)]\right\}. \qquad (2.16)$$

It follows from (2.4), (2.5) that

$$\varphi_1 = (1 + w_2)\vartheta_1 + \left.\frac{\partial\theta_1}{\partial\xi}\right|_{\xi=0},$$
$$f_2 = w_2 + \psi_2 + \left.\frac{\partial\Psi_2}{\partial\xi}\right|_{\xi=0}, \qquad \varphi_2 = \vartheta_2 + \left.\frac{\partial\theta_2}{\partial\xi}\right|_{\xi=0}.$$

The derivatives in these expressions can be calculated from (2.16), (2.13), and (2.14). We then obtain the following three algebraic equations connecting the nine quantities: the constant components, the amplitudes of the first two harmonics of the gradient, the surface temperature, and the combustion velocity

$$\varphi_1 = \vartheta_1(1 + w_2) + A_1z_1 + \left(A_3 - \frac{i\bar{v}_1v_2}{4\gamma^2}\right)z_1 -$$
$$- \frac{A_1w_2z_1}{1 + 2z_1} - \frac{A_1(1 + \bar{z}_1)(\bar{z}_1 - z_1)}{4\gamma} \times$$
$$\times \left[v_2 + \frac{v_1^2}{2\gamma}(1 + \bar{z}_1)\right] + \frac{\bar{v}_1}{2\gamma}(1 + z_2)(z_2 - z_1) \times$$
$$\times \left[A_2 - \frac{v_1A_1}{2\gamma}(1 + z_1) - \frac{i}{4\gamma^2}v_1^2\right], \qquad (2.17)$$

$$f_2 = w_2 + \psi_2 + \frac{1}{4}(\bar\vartheta_1v_1 + \vartheta_1\bar{v}_1), \qquad (2.18)$$

$$\varphi_2 = \vartheta_2 + \frac{iv_1A_1}{2} + z_2\left[A_2 - \frac{iv_1^2}{4\gamma^2} - \frac{v_1A_1}{2\gamma}(1 + z_1)\right]. \quad (2.19)$$

A further six equations can be obtained by expanding the functions $v(\varphi, \eta)$ and $\vartheta(\varphi, \eta)$ in Taylor series up to the third order terms. After multiplication of the trigonometric functions and equating coefficients of like harmonics, we obtain,

$$v_1 = \frac{\partial v}{\partial\varphi}\varphi_1 + \frac{\partial v}{\partial\eta}\eta_1 + \frac{\partial^2 v}{\partial\varphi^2}\left(\varphi_1f_2 + \frac{i\bar\varphi_1\varphi_2}{2}\right) +$$
$$+ \frac{\partial^2 v}{\partial\varphi\,\partial\eta}\left(\eta_1f_2 + \frac{i\bar\eta_1\varphi_2}{2}\right) + \frac{1}{8}\left[\frac{\partial^3 v}{\partial\varphi^3}\varphi_1^2\bar\varphi_1 + \frac{\partial^3 v}{\partial\varphi^2\,\partial\eta} \times\right.$$
$$\left. \times (2\varphi_1\bar\varphi_1\eta_1 + \varphi_1^2\bar\eta_1) + \frac{\partial^3 v}{\partial\varphi\,\partial\eta^2}(2\varphi_1\eta_1\bar\eta_1 + \bar\varphi_1\eta_1^2)\right] \quad (2.20)$$

$$w_2 = \frac{\partial v}{\partial\varphi}f_2 + \frac{1}{4}\left[\frac{\partial^2 v}{\partial\varphi^2}\varphi_1\bar\varphi_1 +\right.$$

$$+ \frac{\partial^2 v}{\partial \varphi \partial \eta} (\varphi_1 \bar{\eta}_1 + \bar{\varphi}_1 \eta_1) + \frac{\partial^2 v}{\partial \eta^2} \eta_1 \bar{\eta}_1 \Big] , \qquad (2.21)$$

$$v_2 = \frac{\partial v}{\partial \varphi} \varphi_2 - \frac{i}{4} \Big[\frac{\partial^2 v}{\partial \varphi^2} \varphi_1^2 +$$

$$+ 2 \frac{\partial^2 v}{\partial \varphi \partial \eta} \varphi_1 \eta_1 + \frac{\partial^2 v}{\partial \eta^2} \eta_1^2 \Big] \qquad (2.22)$$

for the combustion velocity where $\eta_1 = h(1 + i)$. For ϑ_1, ψ_2 and ϑ_2 we obtain expressions of the same form but with the derivatives of v replaced by derivatives of ϑ with respect to φ and η.

The above nine equations can be used to find the constant terms and the first two harmonics w_2, v_1, and v_2 of the combustion velocity, the temperatures ψ_2, ϑ_1, and ϑ_2, and the gradient f_2, φ_1, and φ_2 on the surface of the powder. The solution of these algebraic equations is simpler than the method used to solve Eqs. (2.7)–(2.9). First the linear approximations for θ_1, φ_1, and v_1 are obtained. Then we can find the constant components and the second order approximations for the second harmonics, and finally we obtain the third order approximation to the first harmonic.

3. RESONANCE

It was shown in [4] that powders having a uniform surface temperature represent an oscillatory system with natural frequency γ_0 and damping decrement λ given by

$$\gamma_0 = \frac{\sqrt{k}}{r}, \qquad \lambda = \frac{r(k+1) - (k-1)^2}{2r^2} \qquad (3.1)$$

where k and r are parameters of the powder characterizing the dependence of combustion velocity and surface temperature on the initial temperature, i.e.,

$$k = (T_s{}^\circ - T_0) \Big(\frac{\partial \ln u^\circ}{\partial T_0} \Big)_p, \qquad r = \Big(\frac{\partial T_s{}^\circ}{\partial T_0} \Big)_p. \qquad (3.2)$$

If the frequency γ of the applied pressure is close to the natural frequency of the powder ($\gamma = \gamma_0 + \varepsilon$, $\varepsilon \ll \gamma_0$) and the damping decrement $\lambda \ll \gamma_0$ (the combustion regime is close to the limit of stability $r = (k - 1)^2/(k + 1)$), then the linear approximation to the frequency dependence of the first harmonic of the combustion velocity has the following resonance character [4],

$$v_1(\varepsilon - i\lambda) = f(k, \nu, \mu)\eta_1. \qquad (3.3)$$

The amplitude of the combustion velocity depends on the parameters

$$\nu = \Big(\frac{\partial \ln u^\circ}{\partial \ln p} \Big)_{T_0}, \qquad \mu = \frac{1}{T_s{}^\circ - T_0} \Big(\frac{\partial T_s{}^\circ}{\partial \ln p} \Big)_{T_0} \qquad (3.4)$$

which reveal a rather marked change of combustion velocity and surface temperature with change of pressure.

Now let us examine a nonlinear approximation to resonance. As is known in the theory of mechanical and electrical nonlinear vibrations [5, 6, 7] in the case of nonlinear resonance, the first harmonic amplitude is of the order to the cube root of the forcing force

amplitude, and the constant term and second harmonic are of the order of the square of the first harmonic amplitude. Consequently equations (2.20)–(2.22) and their analogous equations for the surface temperature can be simplified. Indeed, in taking second- and third-order derivatives we must omit terms containing η_1, such as $\eta_1 f_2 \sim \eta_1 v_1^2 \sim \eta_1^{5/3}$ as well as terms of the type $\varphi_1 f_2$ of the order $v_1^3 \sim \eta_1$, i.e., of much higher order.

Thus, the relations $v(\varphi, \eta)$ and $\vartheta(\varphi, \eta)$ yield six equations containing the amplitude of pressure only in the first power and in the linear terms

$$v_1 = \frac{\partial v}{\partial \varphi} \varphi_1 + \frac{\partial v}{\partial \eta} \eta_1 + \frac{\partial^2 v}{\partial \varphi^2} \varphi_1 f_2 + \frac{1}{8} \frac{\partial^3 v}{\partial \varphi^3} \varphi_1^2 \bar{\varphi}_1 ,$$

$$\vartheta_1 = \frac{\partial \vartheta}{\partial \varphi} \varphi_1 + \frac{\partial \vartheta}{\partial \eta} \eta_1 + \frac{\partial^2 \vartheta}{\partial \varphi^2} \varphi_1 f_2 + \frac{1}{8} \frac{\partial^3 \vartheta}{\partial \varphi^3} \varphi_1^2 \bar{\varphi}_1 , \quad (3.5)$$

$$w_2 = \frac{\partial v}{\partial \varphi} f_2 + \frac{1}{4} \frac{\partial^2 v}{\partial \varphi^2} \varphi_1 \bar{\varphi}_1,$$

$$\psi_2 = \frac{\partial \vartheta}{\partial \varphi} f_2 + \frac{1}{4} \frac{\partial^2 \vartheta}{\partial \varphi^2} \varphi_1 \bar{\varphi}_1 , \qquad (3.6)$$

$$v_2 = \frac{\partial v}{\partial \varphi} \varphi_2 - \frac{i}{4} \frac{\partial^2 v}{\partial \varphi^2} \varphi_1^2,$$

$$\vartheta_2 = \frac{\partial \vartheta}{\partial \varphi} \varphi_2 - \frac{i}{4} \frac{\partial^2 \vartheta}{\partial \varphi^2} \varphi_1^2. \qquad (3.7)$$

Relations (2.17)–(2.19) obtained from the heat conduction equation obviously remain unchanged.

Now let us derive the equation for the amplitude of the first harmonic of the combustion velocity from the above equations. In the linear approximation we have

$$\varphi_1 = \vartheta_1 + A_1 z_1,$$

$$v_1 = \frac{\partial v}{\partial \varphi} \varphi_1 + \frac{\partial v}{\partial \eta} \eta_1, \qquad \vartheta_1 = \frac{\partial \vartheta}{\partial \varphi} \varphi_1 + \frac{\partial \vartheta}{\partial \eta} \eta_1 . \quad (3.8)$$

We will express the constant component w_2 of the combustion velocity, the surface temperature ψ_2, and the gradient f_2 in terms of v_1. To this end we substitute the linear approximations to ϑ_1 and φ_1 into Eqs. (2.18) and (3.6).

The last two equations in (3.8) yield

$$\varphi_1 = \Big(\frac{\partial v}{\partial \varphi} \Big)^{-1} v_1, \qquad \vartheta_1 = \Big(\frac{\partial v}{\partial \varphi} \Big)^{-1} \frac{\partial \vartheta}{\partial \varphi} v_1 .$$

The terms containing the pressure amplitude need not be retained since in the second approximation they reduce to expressions of the type $v_1 \eta_1 \sim \eta_1^{4/3}$ which can be neglected in comparison with $v_1^2 \sim \eta_1^{2/3}$. Thus, (2.18) and (3.6) yield three first-order algebraic equations for w_2, ψ_2, and f_2 containing terms with $|v_1|^2$. The solution for these equations shows that each constant component is proportional to the square of the modulus of the amplitude of the first harmonic, i.e., $|v_1|^2$.

The analogous procedure applied to the equations for the second harmonics (2.19) and (3.7) shows that the second harmonic of the combustion velocity v_2, the temperature ϑ_2 and the gradient φ_2 on the surface of the powder are proportional to the square of the amplitude of the first harmonic of the combustion velocity, i.e., v_1^2.

In Eqs. (2.17) and (3.5) for the first harmonic the constant components are all multiplied by the amplitudes of the first harmonics and the second harmonics are multiplied by the complex conjugate amplitudes of the first harmonics. Thus, all third-order terms will be proportional to $v_1|v_1|^2$.

Thus, (2.17) and (3.5) yield

$$\varphi_1 = \vartheta_1 + A_1 z_1 + C_1 v_1 |v_1|^2,$$

$$v_1 = \frac{\partial v}{\partial \varphi}\, \varphi_1 + \frac{\partial v}{\partial \eta}\, \eta_1 + C_2 v_1 |v_1|^2,$$

$$\vartheta_1 = \frac{\partial \vartheta}{\partial \varphi}\, \varphi_1 + \frac{\partial \vartheta}{\partial \eta}\, \eta_1 + C_3 v_1 |v_1|^2, \qquad (3.9)$$

where the coefficients C_1, C_2 and C_3 depend on the frequency and the derivatives of v with respect to ϑ and φ. Assuming ε and λ are small, it is possible (see [4]) to change from r and γ to λ and ε through the relations

$$r = \frac{(k-1)^2}{k+1}\left[1 + 2\left(\frac{k-1}{k+1}\right)^2 \lambda\right]$$

$$\gamma = \frac{\sqrt{k}\,(k+1)}{(k-1)^2}\left[1 + \frac{(k-1)^2}{\sqrt{k}\,(k+1)}\,\varepsilon - 2\left(\frac{k-1}{k+1}\right)^2 \lambda\right]. \quad (3.10)$$

In the third-order terms, γ must be replaced by the following resonance frequency on the stability boundary $(\lambda = 0)$

$$\gamma_0 = \frac{\sqrt{k}\,(k+1)}{(k-1)^2}.$$

Now we have the following equation for the amplitude of the first harmonic

$$v_1\left[\varepsilon - i\lambda + (\alpha_1 - i\alpha_2)\,|v_1|^2\right] = f\,(k,\,v,\,\mu)\eta_1 \quad (3.11)$$

where α_1 and α_2 are functions dependent on k and whose forms depend on the normal derivatives of the combustion velocity and surface temperature, and

$$f\,(k, v, \mu) =$$

$$= \left[(k+1)(k^2+6k+1)\{v\,(k^2-1) + \right.$$

$$+ \left. [v\,(k-1)^2 - \mu k\,(k+1)]\,(1 + i\,\sqrt{k})\}\right] \times$$

$$\times \left[2\,(k-1)^4\,[-\,\sqrt{k}\,(k-1) + i\,(3k+1)]\right]^{-1} \quad (3.12)$$

is the same function as in the linear approximation (3.3).

By taking the modulus of expression (3.11) we obtain the equation of the resonance curve relating the amplitude of the first harmonic to frequency for prescribed damping decrement λ and pressure amplitude H, i.e.,

$$|v_1|^2\left[(\varepsilon + \alpha_1|v_1|^2)^2 + (\lambda + \alpha_2|v_1|^2)^2\right] = |f|^2 H^2. \quad (3.13)$$

Thus, consideration of nonlinear effects leads to a relation between frequency and damping factor at resonance on the amplitude of vibration. As in the case of ordinary nonlinear vibrations of electrical or mechanical systems the correction to the frequency is proportional to the square of the amplitude of vibration. An important feature of the theory is the presence in (3.13) of a correction to the damping factor which depends on the square of the amplitude. This fact could lead to autonomous oscillation: when α_2 is negative the damping factor decreases with amplitude and when $\lambda + \alpha_2|v_1|^2 = 0$, it is possible to have self sustained oscillations, i.e., without any variations in the applied pressure. On the other hand, if the steady-state combustion of the powder is unstable ($\lambda < 0$), self sustained oscillations would be possible under the same conditions with $\alpha_2 > 0$. In the following it will be assumed that $\lambda > 0$. The consideration of the contrary case is analogous. The study of the resonance curves is simplified by the following new variables

$$V^2 = \frac{|\alpha_2|\,|v_1|^2}{\lambda}, \qquad \Omega = \frac{\varepsilon}{\lambda}, \qquad F^2 = \frac{|f|^2\,|\alpha_2|\,H^2}{\lambda^3};$$

then

$$(\Omega + qV^2)^2 + (1 \pm V^2)^2 = \frac{F^2}{V^2} \qquad \left(q = \frac{\alpha_1}{|\alpha_2|}\right) \quad (3.14)$$

where the plus sign refers to a positive value of α_2 and the minus sign refers to a negative value of this coefficient. From (3.14), the dependence of frequency on amplitude of vibration is

$$\Omega = -qV^2 \pm \sqrt{F^2/V^2 - (1 \pm V^2)^2}. \quad (3.15)$$

The sign of q is of no great significance, since changing its sign, which is equivalent to changing that of α_1, has the same effect as changing the sign of the frequency, i.e., the resonance curve is reflected about the vertical axis. In the following it will be assumed that q is positive.

First let us consider the case where α_2 is positive (the damping increases with amplitude). If for small pressure amplitudes we neglect powers of V^2 in (3.14), we obtain the following first approximation for resonance

$$V^2\,(1 + \Omega^2) = F^2. \quad (3.16)$$

The amplitude is a maximum when at zero frequency. An increase in pressure amplitude leads to a displacement of the frequency at which the amplitude of oscillation is a maximum. From (3.14) we have

$$\frac{dV^2}{d\Omega} = \frac{-2V^2\,(\Omega + qV^2)}{3\,(q^2+1)\,V^4 + 4\,(q\Omega+1)\,V^2 + \Omega^2 + 1}. \quad (3.17)$$

The maximum amplitude occurs when

$$\Omega_e = -\,qV_e^2. \quad (3.18)$$

Substitution of this equation into (3.14) yields the following relation between the maximum amplitude of vibration and the given pressure amplitude,

$$V_e^2\,(1 + V_e^2)^2 = F^2. \quad (3.19)$$

The derivative (3.17) becomes infinite when

$$3 (q^2 + 1) V_i^4 + 4 (q\Omega_i + 1) V_i^2 + \Omega_i + 1 = 0 . \quad (3.20)$$

The solution of this equation together with (3.14) yields the coordinates of points on the amplitude curve having an infinite gradient for given values of F and q. We will determine under what conditions such points exist. (These points are of importance when studying the stability of states of oscillation.) Eq. (3.20) is quadratic in V_i^2 and has two equal positive roots when

$$\Omega_i^2 (q^2 - 3) + 8q\Omega_i - 3q^2 + 1 = 0,$$

$$q\Omega_i + 1 < 0 .$$

Moreover,

$$V_i^{*2} = \frac{2}{\sqrt{3}(q - \sqrt{3})}, \qquad \Omega_i = -\frac{4q + \sqrt{3}(q^2 + 1)}{q^2 - 3}. \quad (3.21)$$

Thus, points with infinite radiant exist only when $q > (3)^{1/2}$, i.e., only for sufficiently large ratios α_1/α_2. For a prescribed value of q, such points occur only when the pressure amplitude exceeds the critical value

$$F_i^{'2} = \frac{8 (q^2 + 1)}{3 \sqrt{3}(q - \sqrt{3})^3} \quad (3.22)$$

which can be obtained by substituting (3.21) into (3.14). When $q < (3)^{1/2}$ the resonance curve has no point of infinite gradient irrespective of the pressure amplitude.

Figure 1 shows the resonance curve for q = 10. The curves 1–4 are equal to 0.05, 0.2, 0.4, and 0.8, respectively for values of F^2. Curve e is the locus of maxima on the resonance curves, and the points of infinite gradient lie on the curve i. The critical value of the square of the pressure amplitude is $F_i^2 = 0.227$.

Figure 2 shows the resonance curve for q = 1. The curves 1, 2, 3 correspond to values of F^2 equal to 0.25, 1.00, and 4.00, respectively.
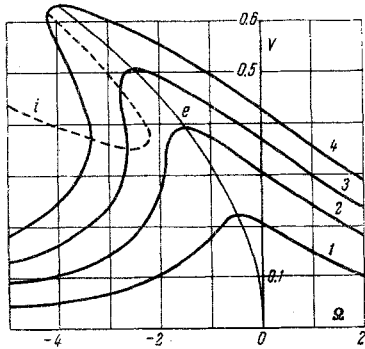

Fig. 1

It is worth comparing the obtained results with the usual nonlinear resonance results. In the nonlinear vibration of electrical and mechanical systems, the damping coefficient does not change with the amplitude of oscillation. Thus, $\alpha_2 = 0$ and $q = \infty$, i.e., resonance of the type shown in Fig. 1 occurs, and for sufficiently large amplitudes of the driving force the resonance curve has two points with infinite radiance and the segmented curve between these points corresponds to unstable oscillations. In the case of the combustion of powders, the damping coefficient changes with the amplitude. When $\alpha_1/\alpha_2 < (3)^{1/2}$, the broadening of the resonance curve with increasing amplitude compensates for its distortion, and there are no points with infinite gradient. For the inverse relation between α_1 and α_2 (strong slope and gentle broadening) there are such points. The value $q = \infty$ corresponds to the case where the damping coefficient does not decrease with amplitude and the above formulas reduce to the well-known expression in the theory of ordinary nonlinear vibrations.
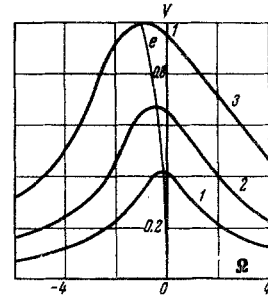

Fig. 2

Now we will consider the resonance curve for $\alpha_2 < 0$, i.e., the case when the damping coefficient decreases with the amplitude of vibration. The extremal value of the amplitude can be determined from the relations

$$\Omega_e = -qV_e^2, \qquad V_e^2 (1 - V_e^2)^2 = F^2. \quad (3.23)$$

The second relation can, depending on the value of F, have either one or three positive roots V_e^2. An elementary investigation shows that for small values of the pressure amplitude,

$$F < F_s, \qquad F_s^2 = {}^4/_{27} \quad (3.24)$$

and there are three roots. When $F > F_s$ there is only one root.

The coordinates of points with infinite gradient can be found from the solution of the following equations,

$$(\Omega_i + qV_i^2)^2 + (1 - V_i^2)^2 = F^2 / V_i^2$$

$$3 (q^2 + 1) V_i^4 + 4(q\Omega_i - 1) V_i^2 + \Omega_i^2 + 1 = 0 . \quad (3.25)$$

When $q^2 > 3$, the second equation has a graph with branches going to infinity. The minimal value of Ω_i and the corresponding values of V_i^2 and F_i^2 are

$$\Omega_i^* = \frac{4q - \sqrt{3}(q^2 + 1)}{q^2 - 3},$$

$$V_i^{*2} = \frac{2}{\sqrt{3}(q + \sqrt{3})}, \qquad F_i^2 = \frac{8(q^2 + 1)}{3 \sqrt{3}(q + \sqrt{3})^3} . \quad (3.26)$$

When $q^2 < 3$, the curve of $V_i(\Omega_i)$ is closed. The extreme

values of Ω_i and the corresponding values of V_i^2 and F_i^2 are

$$\Omega_i^* = \frac{4q \mp \sqrt{3}\,(q^2 + 1)}{q^2 - 3},$$

$$V_i^{*2} = \frac{2}{\sqrt{3}\,(\sqrt{3} \pm q)}, \qquad F_i^2 = \frac{8\,(q^2 + 1)}{3\,\sqrt{3}\,(\sqrt{3} \pm q)^3}. \quad (3.27)$$

Figures 3 and 4 show the resonance curves for $\alpha_2 < 0$. Figure 3 is for $q = 3$ (the values of F^2 on curves 1, 2, 3 are 0.05, 4/27, 0.5, respectively). Curve i passing through points of infinite gradient is open $(q > (3)^{1/2})$. Figure 4 is for $q = 1$ (the values of F^2 on curves 1 through 5 are 0.02, 4/27, 1.00, 4.00, and 8.00, respectively) and the locus
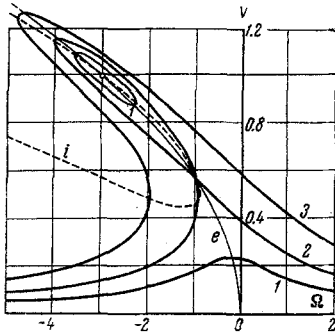

Fig. 3

of points with infinite gradient is a closed curve. In other words, the resonance curves corresponding to large amplitudes of the forcing function will always have two points with infinite gradient provided $q > (3)^{1/2}$. If on the contrary $q < (3)^{1/2}$, there is a critical value of the excitation above which there are no such points on the resonance curve.

The most interesting property of the resonance curves for $\alpha_2 < 0$ is their separation into two branches when $F < F_s = \frac{4}{27}$ (curve 1). The lower branch with small amplitudes of vibration can also be obtained in the linear approximation (for sufficiently small F). On the other hand the closed contour around the point $(-q, 1)$ is a consequence of the nonlinearity of the theory. In fact, when the excitation is absent, the resonance curves degenerate into the line $V = 0$ and the point $(-q, 1)$,
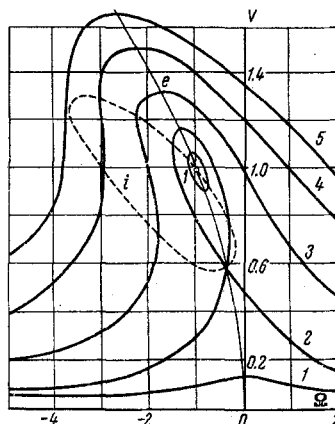

Fig. 4

i. e., the system can execute free vibrations with amplitude and frequency

$$|v_1| = \sqrt{\lambda / |\alpha_2|}, \qquad \varepsilon = -q\lambda.$$

Moreover, the effective damping coefficient vanishes to second-order accuracy in the square of the vibration amplitude. Such a motion of the system is usually called autonomous.

Resonance curves of this type are also encountered in the theory of nonlinear vibrations of electrical systems. In [6], for example, an autonomous system consisting of a valve oscillator with negative feedback is studied. An alternating voltage is injected into the circuit.

An important question in the theory of nonlinear vibrations is the study of the stability of processes on different sections of the resonance curve. This requires a special investigation since mechanical and electrical systems are governed by ordinary differential equations; whereas, we are studying oscillatory processes described by nonlinear partial differential equations and nonlinear restraints between the combustion velocity, the temperature, and gradient on the surface of the powder. The question of self-excited vibrations is connected with the sign of lambda.

4. CALCULATION OF THE COEFFICIENTS α_1 AND α_2

The form of these coefficients is determined by the derivatives up to the third order of the functions $v(\varphi, \eta)$ and $\vartheta(\varphi, \eta)$. There is no point in calculating these coefficients in the general case since this would lead to unwieldy expressions. However, at present there is no theory which would give concrete expressions for the dependence of the combustion velocity and surface temperature on the gradient and pressure. For this reason the coefficients α_1 and α_2 have been computed for particular types of functions $v(\varphi, \eta)$ and $\vartheta(\varphi, \eta)$ which, with a view to application, are similar to the experimental relations.
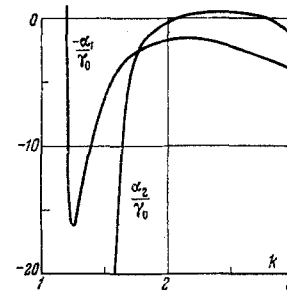

Fig. 5

The experimental data on the combustion velocity of powders in the stationary state can be represented in the form

$$u^o\,(T_0,\,p) = A\,(p^o)^\nu\,e^{\alpha T_0}. \qquad (4.1)$$

Measurements of the surface temperature of the powder for different initial temperatures and pressures have been carried out recently [8]. In view of the unavoidable experimental uncertainty it is not possible to get information on the behavior of the function $T_s^o\,(T_0, p)$ with sufficient accuracy. Therefore the function has been represented in the form

$$T_s{}^o = T_0 + B\,(p^o)^\mu\,e^{\beta T_0}. \qquad (4.2)$$

The parameters B, μ, β can be chosen to fit the experimental data with sufficient accuracy, and the form of this function is convenient for computations. The same applies to (4.1).

With (1.1) and the dimensionless variables we obtain

$$v = \eta^\nu e^{k\,(\vartheta - \varphi/v)}, \qquad \frac{\varphi}{v} = \eta^\mu e^{(r-1)\,(\vartheta - \varphi/v)} \qquad (4.3)$$

where k, r, ν and μ are given in (3.2) and (3.4).

Expansion of these functions in series gives six algebraic equations as in (3.5)–(3.7) in which the particular expressions (4.3) are to be used. The subsequent course of the computations is as follows: one is led to algebraic transformations of the complex functions depending only on the parameter k since in the third-order terms r and γ are to be replaced by $(k-1)^2/(k+1)$ and $(k)^{1/2}(k+1)/(k-1)^2$. The parameters ν and μ enter in the result in the same way as in the linear approximation (3.11), (3.12). Simple but laborious calculations lead to

extremely unwieldy expressions for $\alpha_1(k)$ and $\alpha_2(k)$. It has been shown that each coefficient can be written as a fraction, whose numerator and the denominator are the sums of products of various powers of k

$$R_2{}^\circ = \left[\frac{1}{8} \left(64 \frac{k\,(k+1)^2}{(k-1)^4} - 1 \right)^{1/2} + 1 \right]^{1/2}$$

and polynomials of order six and ten in k. The results of the numerical calculations are shown in Fig. 5 in the form of graphs of $(-\alpha_1)/\gamma_0$ and α_2/γ_0 versus k. When $k - 1 \ll 1$, we have the asymptotic expressions

$$\frac{\alpha_1}{\gamma_0} = -\frac{0.334}{(k-1)^4}, \qquad \frac{\alpha_2}{\gamma_0} = \frac{1.94}{(k-1)^4}. \qquad (4.4)$$

The graphs show that the correction to the damping decrement $\alpha_2 |v_j|^2$ can be either positive or negative. When $\alpha_2 > 0$, we have $q > (3)^{1/2}$. Thus for the present model of the powder represented by relations (4.3), either of the resonance curves in Figs. 1, 3, and 4 is possible.

The parameters k and r can be varied in experiments by means of heating or cooling of the powder. Each value of the initial temperature T_0 of the powder corresponds to a point in the kr-plane. Experiments [8] show that an increase in T_0 leads (in the case of powder H) to an increase in the parameters k and r, and for sufficiently high temperatures the combustion occurs close to the stability limit $r(k+1) = (k-1)^2$. Thus it is to be expected that autonomous oscillations could be encountered in combustion experiments. In [8] it was shown that the combustion of powder H at pressure p = 1 atm and $T_0 < 50°$ C was accompanied by pulsations at a frequency of about one hertz. This corresponds with the expected frequency of autonomous oscillations. In fact, the frequency of vibration is of the order $(u^\circ)^2/\varkappa$. When $\varkappa \sim 10^{-3}$ cm^2/sec and $u^\circ \sim 5 \cdot 10^{-2}$ cm/sec, we have $\gamma \sim 2$ sec^{-1}.

The above reasoning applies to the case k > 1, i.e., to the region in which the combustion can execute free oscillations. However, the combustion can exhibit autonomous oscillations also when k < 1. The nonlinear effects can cause the damping decrement to decrease and vanish even if small oscillations are not possible. This can be illustrated by a simple pendulum suspended in a medium whose viscosity decreases with height. If the viscosity about the equilibrium position is sufficiently large, small oscillations about this position will be aperiodic. However, for sufficiently large amplitudes the pendulum could move in regions of negative damping. In this case it is possible to have large undamped free oscillations, the pendulum gaining energy in the region of negative viscosity and dissipating it about the equilibrium position. Correspondingly, it can be shown that the powder combustion with k < 1 can also exhibit the same behavior.

In conclusion it is to be noted that, as in any nonlinear system, the powder must exhibit resonance for frequencies of pressure variation equal to $\gamma_0 p/q$ where p and q are integers. These ultra- and sonic oscillations are well-known in the theory of electrical and mechanical oscillations. The method, explained in the present paper, enables one also to study these effects in the combustion processes of powders.

REFERENCES

1. B. V. Novozhilov, "Stability criterion for the steady-state combustion of powder," PMTF, [Journal of Applied Mechanics and Technical Physics], no. 4, 1965.

2. Ya. B. Zel'dovich, "Theory of combustion of powders and explosive substances," Zh. eksperim. i teor. fiz., vol. 12, no. 11–12, 1942.

3. Ya. B. Zel'dovich, "Combustion velocity of powder under variable pressure," PMTF, no. 3, 1964.

4. B. V. Novozhilov, "Combustion of powder under a harmonically varying pressure," PMTF [Journal of Applied Mechanics and Technical Physics], no. 6, 1965.

5. L. D. Landau and E. M. Lifshits, Mechanics [in Russian], Fizmatgiz, 1958.

6. J. Stoker, Nonlinear Oscillations in Mechanical and Electrical Systems [Russian translation], IIL, 1952.

7. A. A. Andronov, A. A. Vitt, and S. E. Khaikin, Theory of Oscillations [in Russian], Fizmatgiz, 1959.

8. A. A. Zenin, O. I. Leipunskii, A. D. Margolin, O. I. Nefedov, and P. F. Pokhil, "Temperature field at the surface of burning powder and stability of combustion," Dokl. AN SSSR, vol. 169, no. 3, 1966.